Students' Measurement Strategies of Area[1]

ABSTRACT *This is a study of the strategies and area measuring devices used by a sample of 106 students of the last grade of Elementary School. Our research plan includes the comparison between strategies for area measurement, as used by two groups: the experimental group (E.G.) and the control group (C.G.). The experimental group attended a special teaching course, which stressed the conceptual characteristics of the area measurement process. The research question to be answered in the present research is whether the tools available to students for area measurement as well as the special teaching practices 'lead' the students to specific measurement strategies.*

Keywords: Measurement strategies; Area Measurement; Elementary School; Geometry

THEORETICAL REMARKS

A lot of researches on area measurement highlight the problems the students have with understanding measurement processes. Such researches usually put the blame on the traditional way of teaching, which is based on the algorithm Area = base X height or Area = length X width (e.g. Battista, 1982; Nunes et al., 1993; Nitabach & Lehrer, 1996; Outhred & Mitchelmore, 1996; Kidman & Cooper, 1997; Clements & Stephan, 2004). The above researches pay particular attention to the difference between the process of length measurement and area measurement. In other words, it is stressed that the length is directly measured, while the area is indirectly measured by using longitudinal quantities appearing in the formula of the area.

The historical perspective embodied in the teaching of mathematical concepts facilitates the understanding of both the importance and the meaning of mathematical concepts and, in particular, Geometry, which is our present objective. Before

[1] Zacharos, K. (2005), Students' Measurement Strategies of Area, *Mediterranean Journal for Research in Mathematics Education*, 4(2), 111-127.

Geometry developed to a theoretical and logical fabrication, it was a human tool determining the relationships between humans and their domain and environment. It started as an attempt to measure quantities such as length and area. However, the development of Mathematics has led non-mathematicians to "look at a mathematics text today, while they might be inclined to describe mathematics as 'all algebra'" (Fowler, 1987, p. 9). This tendency towards arithmetisation in Geometry shows that "the "area" of a rectangle is the product of its base by the height…" (Fowler, 1987, p. 8). However, on studying the history of Mathematics, it can be realised that "Greek Mathematics up to the second century BC seems, to an extraordinary degree, to be different" and "it seems to be completely non-arithmetised" (Fowler, 1987, p. 10).

The main strategy of the Euclidean Geometry in determining equality of quantities is that of *overlapping*. This process may be found in the criteria of equality of triangles and may be expansively used in determining the equality of areas as well, because the speculation developed on areas had nothing to do with the modern algebraic calculating methods, but with 'quality' approaches proper to the process of overlapping (Bunt, et al., 1976). Such approaches include the process of overlapping the measured area with areas selected as measurement units (Battista, 1982; Nunes et al., 1993; Nitabach & Lehrer, 1996).

Furthermore, it is stressed that the process of measurement may be more effective when there is a correspondence between the dimension of the measurement tool and the dimension of the measured area. Thus, the areas are suggested to be measured with two-dimensional units, such as squares, rectangles, etc. (Nunes et al., 1993).

However, the selection of the square area, which has been established as area measurement unit, is a neither obvious nor spontaneous selection of students, when they have not attended a relative teaching course. The shape of the measured area affects the selection of the measurement unit and is usually the same shape as the measured area (Heraud, 1987). The square unit is usually selected for overlapping rectangles and other figures with right angles. In cases of other figures, measurement units selected differ from square units (Maher and Beattys, 1986).

The ability of children to use sign systems is not an obvious consequence of their thought development. Many intervening sign systems are not discovered by children but are due to some social mediation. When children become familiar with a sign system, the sign system greatly affects the structure of their thought (Nunes, 1997). According to T. Nunes (1997) and as regards area measurement of plane figures, the measurement tools provided each time play a structural role in becoming familiar with the concept of area. Because "the structuring of the children's action was not independent from the tool they had at their disposal in the problem-solving situation" (Nunes, 1997, p. 308).

The research question to be answered in the present research is whether the tools available to students for area measurement as well as the special teaching practices 'lead' the students to specific measurement strategies.

THE SAMPLE AND THE RESEARCH DESIGN

The sample of the study consisted of 106 students attending the last grade of elementary school, approximately 11 years old. In particular, our research planning involves the comparison of performances between two research groups: the *experimental group* (E.G. – three segments with 56 students) and the *control group* (C.G. – three segments with 50 students). The students of both research groups come from the same schools, where classes are arranged in alphabetical order. The element that differentiates the first group from the second is the independent variable. This is introduced to the experimental group in order to evaluate its effects. Our research uses the teaching of the experimental group and the measurement 'tools' used by the experimental group as independent variables. The experimental plan used is the "post test – only control group design" (Cohen & Manion, 1989).

The criterion for the drawing of the sample from the last grade of elementary school was the fact that this is the grade where the teaching of the measurement of areas of plane figures is concluded. In this way we were given the opportunity to assess the knowledge of students who graduate from primary school, and to investigate the strategies followed for the solution of the tasks given to them.

The subjects of the experimental group participated in a teaching session based on a model teaching unit given to the teacher of the class. With the plan of lesson we proposed, we tried to focus on the conceptual characteristics of area measurement. More specifically the teaching referred to the following topics: The first topic has to do with the Euclidian method for the area comparison. It is noted that when Euclid wanted to show that two schemas have equal areas, he proved that one of them could be divided in such parts so as if these are reconstructed properly can create the other schema. This procedure is what is called “additivity axiom” (Wagman, 1975; Freudenthal, 1983). The second has to do with the principle of “overlapping”. The Euclidean geometry uses as a general proof method the principle of “overlapping” (in Euclidean Geometry the term ‘epithesis’-‘επίθεσις’). An expansive interpretation involving area measurement/overlapping is used here. Thus, the area of a surface is the quotient of the division of the evaluated area by the area unit.

Activities related to both the comparison and evaluation of areas are carried out throughout teaching. At this point the logic of either their analysis and reconstruction or overlapping is introduced. In case of overlapping, different shapes, such as rectangles, triangles, squares, trapeziums as well as ‘abnormal’ geometrical shapes are used as measurement units (appendix A). The teaching is an introduction to the concept of area and its measurement.

The students of the control group do not participate in some special teaching course but they are taught the same cognitive object, as defined in the curriculum and instructed by the schoolbook of Mathematics.[1] In particular, when measuring the area of plane figures, emphasis is put on the use of formulas and not on conceptual approaches. This was the case when the subject was taught in C.G., when, for example, during the area measurement of the rectangle, the interest was focused on finding the relevant formula (Area = length X width). The quadrature of the area of the rectangle appearing in this procedure is not a separate instructive strategy but simply a necessary step towards formula reasoning. In all cases the students were taught by their classroom teachers.

The collection and analysis of empirical data of the study included a combination of qualitative and quantitative approaches. In the quantitative approach we include the use of the experimental method and the structured questionnaire (appendix B). The qualitative characteristics of the method are presented in the way we gather empirical data in the form of a “clinical interview” (Hunting, 1997). The answers are recorded with a view to logging all verbal answers and the students’ comments for their further quality evaluation.

Measurement Tools

In the first three tasks (appendix B) the measurement tools used by the E.G. are the square centimeter designed on their worksheet that will be used by the students. They are also given a ruler without numbers in order to draw straight lines more easily. In case there is difficulty in ‘transferring’ the square unit to the measured area, they are given a small cardboard of one square centimeter. The students of the control group have a numbered ruler. In the other task all the students of the sample are given a numbered ruler. The two groups of students worked with pencils so that the errors could be corrected.

Two weeks after the completion of the area measurement teaching course, all the subjects participated in a personal interview, where they were asked to answer in tasks related to area comparison and measurement, which were different from those used in teaching. The interview included teaching elements as well. In other words, each new task was presented after the previous task had been answered, either from the student or with the help of the researcher. The tasks used were taken either from researches evaluating the knowledge of students in the research field we were interested in or were proposed by the researcher. In all cases the symbols and words of the schoolbook were used. We have selected the tasks after empirical tests on students not included in the final research sample in order to further safeguard the structural validity of the tasks proposed.

RESULTS

It is our purpose here to inspect the effect of both the particular teaching practices and measurement tools used by each research group on area measurement. We are interested in one aspect of this effect: the possible differentiation between solution strategies.

The construction of a measurement tool for length

The difficulties of the experimental group subjects either in measuring the sides of the rectangle or reproducing the unit surface led these students to the construction of a meter for measuring distances. Some subjects (approximately 64% of the experimental group subjects) used the unnumbered ruler, put it along one side of the square centimeter and marked the length of one centimeter with a pencil. They then reproduced this length on two successive sides of the figure. Some other students (approximately 27% of the experimental group subjects) used the card of one square centimeter as measurement unit to draw two successive sides of the rectangle.

Analysis of area measurement strategies use by task

Then we record the measurement strategies used by the students of both research groups in tasks 1-4 (appendix B). When processing the empirical data, the strategies of quadrature and enumeration as well as the analysis of an area into more familiar shapes, which were teaching objects in the E.G., belonged in the same category. On the other hand, the direct use of formulas in all cases was a second category itself.

First Task

In the first task (fig. 1, appendix B), students were asked to calculate the area of a rectangle 4cmX6cm. Apart from the designed rectangle, the square centimeter was designed on the worksheet as well.

As for the rectangle measurement strategies, we discerned the following basic strategies: First, the strategy of surface quadrature and enumeration, used mainly by the students of the experimental group (fig. 1). This process is occasionally a process of counting imaginary squares, as it happened in the case of a student resorting to a thinkable enumeration of the squares: “four and other four, eight. Eight and four is twelve… and other four is twenty-four”. It is a procedure of unitizing as this is analyzed by Wheatley & Reynolds (1996).

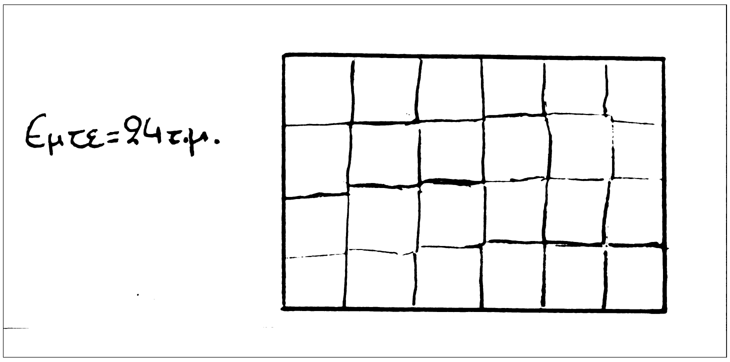

Figure 1. Overlapping by Using the Square Centimeter (Area=24cm^2)

The second strategy uses the formula. It was mainly followed by control group students, who used the ruler to measure two successive sides of the rectangle and multiply, as well as by students that quadrated the area and then resorted to the formula. For example, a student counts horizontally six imaginary squares and four vertically. During the multiplication, the imaginary angular square (fig. 2) is measured only once and the result is: Area = 6X3 = $18cm^2$. This is a common mistake found even in the prospective teachers and is being mentioned by Simon (1995).

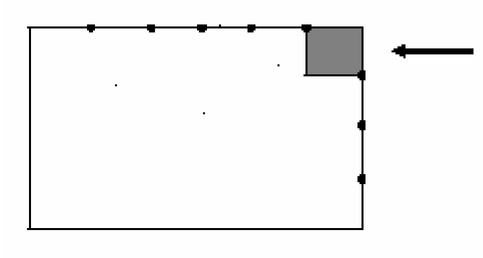

Figure 2. The use of the angular square in the formula base X height

The data of table shows that there is an actual difference, which is statistically significant, between the strategies selected by the two research groups (X^2=35.196, p=0.000).

Table

The prevailing measurement strategies of the two research groups in tasks 1-4

		E.G.		C.G.	
		N	F%	N	F%
Figure 1	*Quadrating and Enumeration*	30	53.6	1	2
	Use of Formulas	23	41.1	47	94

Figure 2	*Quadrating and Enumeration*	40	71.4	10	20
	Use of Formulas	5	8.9	19	38
Figure 3	*Quadrating and Enumeration*	36	64.3	2	4
	Use of Formulas	7	12.5	18	36
Figure 4	*Quadrating and Enumeration*	31	55.4	14	28
	Use of Formulas	10	17.9	16	32

Second Task

In the second task (fig. 2, appendix B), the students with correct answers mainly follow two strategies: They either quadrate the area of the shape and enumerate the squares or divide the original shape into rectangles and then add up the areas of the rectangles (71.4% of E.G. subjects and 20% of C.G.). On the other hand, the students that fail either resort to the use of formulas (mainly the rectangle formula, 8.9% of E.G. and 38% of C.G.), or claim that the shape has no area or there is no way to calculate it (one E.G. student and nine C.G. students), because the 'shape is odd' or '…it has too many lines and too many sides'!

For example, after a student having quadrated the area of the figure with the help of the square card, it used the multiplication 5X4=20 cm^2 (5 are the squares touching the base and 4 are the remaining squares). An extract of the dialogue follows:

R (researcher): What is its area?

S52: Twenty.

R: How many squares (like the square card he uses) fit on the figure's surface?

S52: Four plus five, nine.

R: What is the figure's area, finally? Nine or twenty?

S52: Oh! I'm mixed up!…Twenty…Nine…Nine squares fit, but if we multiply, they are twenty!

Finally, there are twelve C.G. students (24% of C.G.) that state they ignore the matter and give up.

The differences between measurement strategies selected by the two research groups in the second task (table) are statistically significant ($X^2 = 23.821$, $p = 0.000$).

Third Task

In the third task (fig. 3, appendix B) we also note the strategies of quadrature and enumeration, however not always successfully, since some students enumerate half of the square units (the triangles formed) as whole ones. Moreover, some other students analyze the shape into familiar shapes (such as square and rectangular triangle) and proceed with calculation. Finally, in the strategies of formula use, we mainly note the use of the rectangular and trapezium formulas. The most common way is the multiplication 4X2. In many cases, while the area was quadrated, the students resorted to the use of the formula in lots of versions as regards the quantities of base and height. For example, S2 (C.G.) found that Area=3.5X2=7 square centimetres, with 3.5 = the square units touching the base of the figure (fig. 3).

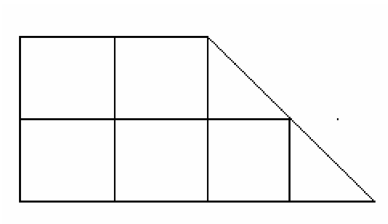

Figure 3. Area = 3.5X2

The data in table shows that the differentiation between the two groups as regards measurement strategies is statistically significant (X^2 = 30.995, p = 0.000).
In the fourth and fifth tasks (fig. 4 and 5, appendix B) the only measurement tool the students of both research groups have is the numbered ruler.

Fourth Task

In the fourth task a percentage of 55.4% of E.G. students prefer the strategies of quadrature, analysis and reconstruction of the shape's area; these matters were the objects of teaching as well, while the respective percentage in C.G. students was 28%. As regards the use of the formulas, approximately 17.9% of E.G. students and 32% of C.G. students use the equation E = 3X3.

Finally, 26.7% of E.G. students and 40% of C.G. students say that they either do not know ways of calculating the area or the specific shape has no area because 'it is an odd shape'!
The data of students' strategies (table) shows that the discrepancy between the two groups is statistically significant (X^2 = 6.253, p = 0.012).

CONCLUSION

One of the most important issues this research demonstrates is how the tools available to students for area measurement as well as the special teaching practices ‘lead’ the students to specific measurement strategies. A general principle that affected our teaching attempt was that the intervening measurement unit should have the same dimensions as the measured quantities; otherwise the measurement tool used should preserve the natural characteristics of the measured quantity. The use of two-dimensional units is more effective as regards teaching in area measurement, as the case is in length measurement, where longitudinal measurement units are used. We should underline here that the measurement tools used are neither obvious results of the cognitive development of students nor univocal derivatives of the way their thoughts are constituted, but the outcome of a social mediation (Nunes, 1997; Vygotsky, 1978) and, as regards teaching concerning the present research, they result from an intentional and systematic teaching attempt. A correlation between measurement tools available to students and strategies selected for area measurement was noted as well.

The common teaching approach and mathematics textbooks emphasize the quantitative aspects of measurement of area, which is based on the algorithms. One may argue that this approach is inappropriate for students understanding the conceptual characteristics of area measurement. In contrast, an alternative way of teaching suggests enhancing the qualitative approach in the process of measurement (Brown, 2001; Kidman & Cooper, 1997; Outhred & Mitchelmore, 1996).

Although the process of overlapping is considered as a particularly successful strategy for area measurement of polygonal figures, it is neither always an effective strategy nor the only method of measurement. For example, although the use of the square unit area approximates the area of the circular disk, the teaching approach by using normal polygons may be preferable (Freudenthal, 1983).

In conclusion, an issue for future research would be how it may be possible for teacher educators to provide alternative teaching approaches so that students have a better understanding the process of the area measurement.

[1] In the Greek educational system schoolbooks are approved by the Pedagogical Institute and are the same for all students.

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APPENDIX A

(Activities related to comparison and evaluation of areas which are carried out throughout teaching)

(I). The Euclidean method of Area Comparison.

Students were asked to compare the areas of the figures 1a, 1b and 1c.

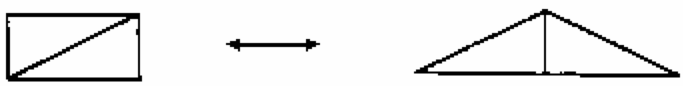


Figure 1a

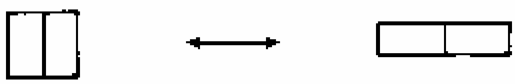


Figure 1b

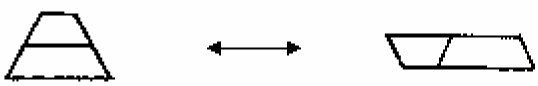


Figure 1c

(II). The Principle of 'Overlapping'.

Students' were asked to measure the areas of the figures 2a, 2b, 2c and 2d using the given units with the strategy of 'overlapping'.

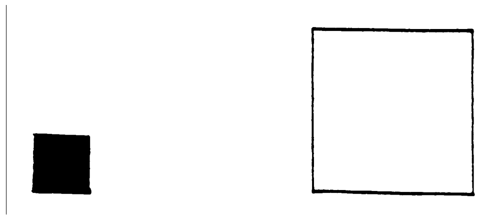


Figure 2a

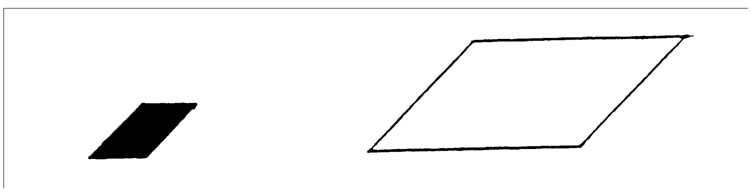
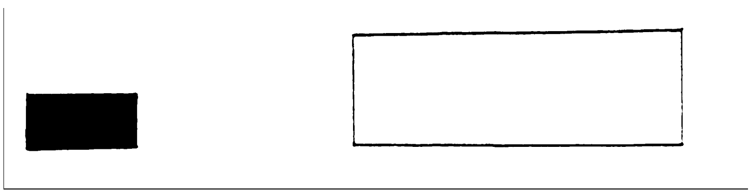


Figure 2b

Figure 2c



APPENDIX B

(The area of the proposed figures is asked to be measured in all tasks)

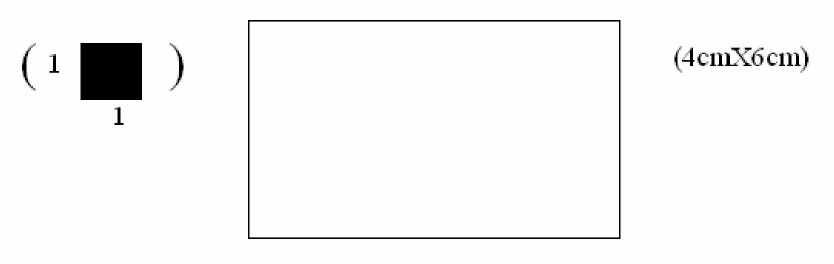


Figure 1 (the square centimeter is drawn on the E.G. worksheet only).

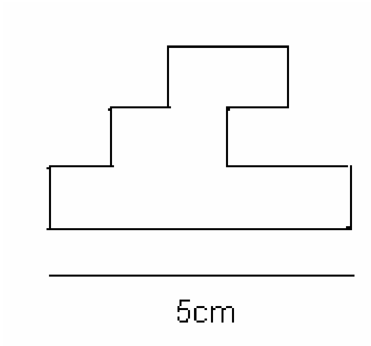


Figure 2

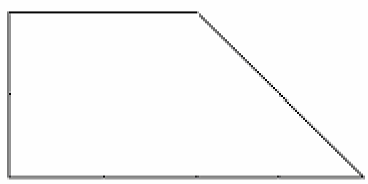


Figure 3

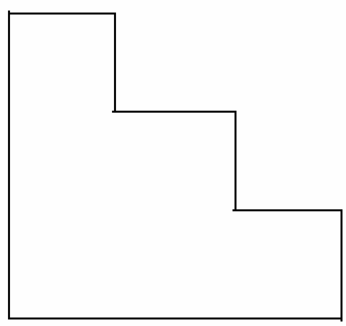


Figure 4